CALCULATION OF THE THIN PLATES WITH THE LARGE DEFLECTIONS

Aliaksei Tarasevich, Victor Tur

Pope John Paul II State School Higher Education in Biala Podlaska,
Department of Engineering Sciences, Department of Civil Engineering
Sidorska Street 95/97, 21-500 Biała Podlaska, Poland
e-mail: vvtur@bstu.by

Summary:
Very large displacement but small strain of very thin plates is studied using Kirchhoff theory.
When plates are deflected beyond a certain magnitude, the linear theory loses it's validity and produces incorrect results. In order for an accurate large deflection solution, one needs to include the coupling between axial and transverse motion, which is geometric non-linearity.

Keywords: thin plates; displacement; deflection.

Introduction

Elasticity theory treats typically linear theory of «rigid» plates. Theory of elasticity in linear version can be used for calculation of plates with small deflection, no more than 1/4…1/5 plate thicklness.

In these plates, deflections normal to the mid surface of the plate are so small that they do not affect the deformation of the element (Samul 1982).

However, thin plates are widely used in various fields (construction, shipbuilding, aircraft manufacturing).

For thin flexible slabs «load-deflection» relationship is non-linear and hypothesis about non-deformability of mid-surface is unfair, because it appears tensile, compression and shear strains.

For large deflection of plates and the appropriate boundary conditions, axial forces in mid-surface appear independently of the effect of horizontal in-plain loads (Umanski 1973).

Usually in the classic theory of elastic thin slabs used Kirchhoff's (Volmir 1956) it is assumed a mid-surface plane can be used to represent the three-dimensional plate (slab) in two-dimensional form. The following main kinematic assumptions that are made in this theory (Volmir 1956):

1. Straight lines normal to mid-surface remain straight after deformation (shear strains are absent $\gamma_{yz} = 0; \gamma_{xz} = 0$).
2. Straight lines normal to mid-surface remain normal to the mid-surface after deformation ($\varepsilon_z = 0$; linear strain in z-direction is absent).
3. Thickness of the plate does not change during deformation ($\sigma_z = 0$) and in mid-surface of plate tensile, compression and shear strains are absent ($U_o = V_o = 0$).

When the thin plates are deflected beyond a certain magnitude, the linear theory loses its validity and produces incorrect results. Linear theory can predict that thedeflection of the member may exceed the length of the member, which is unrealistic. In order for an accurate large deflection solution, one needs to include the coupling between axial and transverse modion (deflection), which is geometric nonlinearity. If the edges of plate are allowed to move freely within the plane of underformed member, this boundary condition is called «stress-free».

If the edges are restricted from moving, the edges require an equivalent axial load to prevent motion, which is called «immovable» boundary condition.

Nonlinear deflection theories also couple axial loads and transverse deflections.

In the mid-surface of thin-plate there are tensile, compressive and shear forces.

From the hypothesis 2:

$$\varepsilon_z = \frac{\partial W}{\partial z} = 0, \tag{1}$$

and deflections of the thin plate does not depend on the coordinates z:

$$W = W(x, y) \tag{2}$$

Thus, all points lying on a vertical line have the same displacement Z. Consequently, to determine the vertical displacements of all point of the plate, sufficient to determine the displacements of its mid-surface.

Strains and curvatures of mid-surface

Using conditions:

$$\left.\begin{array}{l} \gamma_{yz} = 0 \\ \gamma_{xz} = 0 \end{array}\right\} \tag{3}$$

we get:

$$\left.\begin{array}{l} \gamma_{yz} = \dfrac{\partial W}{\partial z} + \dfrac{\partial W}{\partial y} = 0 \\ \gamma_{xz} = \dfrac{\partial W}{\partial x} + \dfrac{\partial W}{\partial z} = 0 \end{array}\right\} \tag{4}$$

and integrating over z, we get the expressions for calculating the displacement of the mid-surface of a deflections:

$$\begin{array}{l} U = U_0 - Z\dfrac{\partial W}{\partial x} \\ V = V_0 - Z\dfrac{\partial W}{\partial y} \end{array}, \tag{5}$$

U_0, V_0 – displacement aline X and Y coordinates axis respectively.

Strains of the points on the mid-surface of plate:

$$\varepsilon_x = \frac{\partial U_0}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 - Z\frac{\partial^2 W}{\partial x^2}$$
$$\varepsilon_y = \frac{\partial V_0}{\partial y} + \frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2 - Z\frac{\partial^2 W}{\partial y^2} \quad (6)$$
$$\gamma_{\mathbf{y}} = \frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} + \frac{\partial W^2}{\partial x \partial y} - 2Z\frac{\partial^2 W}{\partial x \partial y}$$

Curvatures of mid-surface:

$$\chi_x = -\frac{\partial^2 W}{\partial x^2};$$
$$\chi_y = -\frac{\partial^2 W}{\partial y^2}; \quad (7)$$
$$\chi = -\frac{\partial^2 W}{\partial x \partial y}.$$

Expression for strains compatibility:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 W}{\partial x \partial y}\right)^2 - \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}. \quad (8)$$

Stress in thin slabs (plates). Stress-strain relationship.

Stress in the thin plates can be considered as the result of superposition of two states: 1) normal uniformely distributed over the plate thickness; 2) bending stress. Consequently, equations must be written for deformed state of thin plate.

Force projection equation on X and Y axis give expressions:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0;$$
$$\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \quad (9)$$

The equations of moment about the axes X and Y give expressions:

$$\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} = 0;$$
$$\frac{\partial H}{\partial x} + \frac{\partial M_y}{\partial y} = 0. \quad (10)$$

After summing the projections of all forces on the Z-axis for element of plate with sizes *dx*, *dy* and vision by *dxdy* we obtain the following expression:

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\sigma_x h\frac{\partial^2 W}{\partial x^2}+\sigma_y h\frac{\partial^2 W}{\partial y^2}+2\tau h\frac{\partial^2 W}{\partial x\partial y}+q(x,y)=0. \tag{11}$$

Assuming, that the strains of plate are elastic, and normal stress in the direction Z are very small in comparison with the normal stress parallel to the mid-surface of the plate. Following «strain-stress» relationships can be written:

$$\begin{aligned}
&\varepsilon_x=\frac{\sigma_x}{E}-\mu\frac{\sigma_y}{E}; &&\sigma_x=\frac{E}{1-\mu^2}\left(\varepsilon_x+\mu\varepsilon_y\right);\\
&\varepsilon_y=\frac{\sigma_y}{E}-\mu\frac{\sigma_x}{E}; &&\sigma_y=\frac{E}{1-\mu^2}\left(\varepsilon_y+\mu\varepsilon_x\right);\\
&\gamma=\frac{2(1+\mu)}{E}\tau; &&\tau=\frac{E}{2(1+\mu)}\gamma.
\end{aligned} \tag{12}$$

Expression for calculation of bending, torsion moments and shear forces can be written:

$$\begin{aligned}
&M_x=-D\left(\frac{\partial^2 W}{\partial x^2}+\mu\frac{\partial^2 W}{\partial y^2}\right)=-D\left(\chi_x+\chi_y\right);\\
&M_y=-D\left(\frac{\partial^2 W}{\partial y^2}+\mu\frac{\partial^2 W}{\partial x^2}\right)=-D\left(\chi_y+\chi_x\right);\\
&H=-D(1-\mu)\frac{\partial^2 W}{\partial x\partial y}=-D(1-\mu)\chi;\\
&Q_x=-D\left(\frac{\partial^3 W}{\partial x^3}+\frac{\partial^3 W}{\partial x\partial y^2}\right)=-D\frac{\partial}{\partial x}\left(\chi_x+\chi_y\right);\\
&Q_y=-D\left(\frac{\partial^3 W}{\partial y^3}+\frac{\partial^3 W}{\partial y\partial x^2}\right)=-D\frac{\partial}{\partial y}\left(\chi_y+\chi_x\right)
\end{aligned} \tag{13}$$

Basic differential equations

Substituting the expression (13) for the shear forces in the equation of equilibrium (11) and get the following expression:

$$D\nabla^4 W=h\sigma_x\frac{\partial^2 W}{\partial x^2}+h\sigma_y\frac{\partial^2 W}{\partial y^2}+2h\tau\frac{\partial^2 W}{\partial x\partial y}+q. \tag{14}$$

Equation (14) relates the deflection of slab and vertically applied load, but contains additional unknown σ_x,σ_y,τ .

To solve the problem use equations (8) and (12):

$$\frac{\partial^2\sigma_x}{\partial y^2}-2\frac{\partial^2\tau}{\partial x\partial y}+\frac{\partial^2\sigma_y}{\partial x^2}-\mu\left(\frac{\partial^2\sigma_x}{\partial y^2}+2\frac{\partial^2\tau}{\partial x\partial y}+\frac{\partial^2\sigma_y}{\partial x^2}\right)=E\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right] \tag{15}$$

The introduction of the stress function (Eri-function) equations (14) and (15) form a system of non-linear differential equations from theory of flexible plates (Karman-equations) (Konchakovskiy 1984):

$$\begin{aligned}&\frac{D}{h}\nabla^4 W=\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 W}{\partial y^2}-2\frac{\partial^2 F}{\partial y\partial x}\frac{\partial^2 W}{\partial x\partial y}+\frac{1}{h}q;\\&\nabla^4 F=E\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W}{\partial x^2}\right]\end{aligned} \tag{16}$$

System of non-linear differential equations (16) together boundary conditions are basic system of non-linear differential equations flexible plates theory. The solution of system (16) is not obtained in general form, but received a number partial solutions.

Boundary conditions depend on the restricting conditions on the contour of plate.

For example for conditions:

$$\begin{aligned}&U_{x=a}-\;U_{x=0}=0;\\&V_{y=b}-\;V_{y=0}=0.\end{aligned} \tag{17}$$

boundary conditions may be written in the form:

$$\begin{aligned}&\left[\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}-E\left(\frac{\partial W}{\partial x}\right)^2\right]dx=0;\\&\left[\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}-E\left(\frac{\partial W}{\partial y}\right)^2\right]dy=0.\end{aligned} \tag{18}$$

Special computer program was developed for the calculation of thr thin plates under various boundary conditions using the package «MATEMATICA».

Consider the example of calculating the square and rectangular (side ratio 1:2) supported on four corner columns and loaded with concentrate loads in the middle of each finite element (see fig. 1).

Colculations results in the form of diagrams of deflections are shown on the fig. 1 and 2. Using a system of non-linear differential equations linking load and deflection, can be constructed diagrams bending and torsion moments, shear forces and stress.

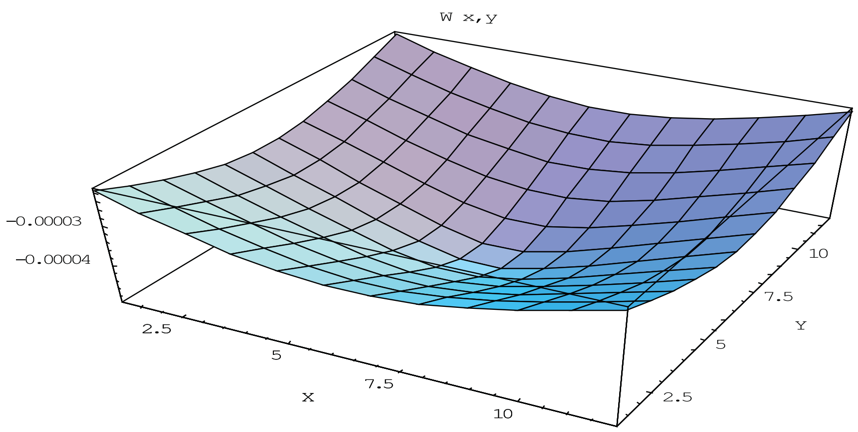

Fig. 1. Deflection of square plate

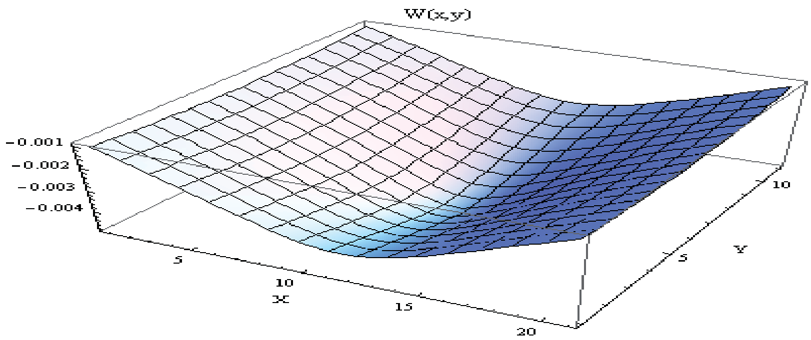

Fig. 2 Deflection of rectangular plate

Conclusion

Analysis of the values of the bending moments and axial forces in thin plate can determine the ratio of bending and membrane stress as a function of deflection. The solutions obtained are in satisfactory agreement with the results of calculating using the formulas of (Umanski 1973).

References:

1. Samul V. (1982), *Principles theory of elasticity and plasticity.* Vyshaja skola, Moscow.
2. Volmir A. (1956), *Flexible plates and membranes.* Gosizdat, Moscow.
3. Konchakovskiy Z (1984), *Slabs. Static design.* Stroyizdat, Moscow.
4. Umanski A. (1973), *Guide of designer.* Stroyizdat, Moscow.